

A SHORT PROOF OF A RESULT OF KATZ AND WEST

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ABSTRACT. We give a short proof of a result due to Katz and West: Let R be a Noetherian ring and I_1, \dots, I_t ideals of R . Let M and N be finitely generated R -modules and $N' \subseteq N$ a submodule. For every fixed $i \geq 0$, the sets $\text{Ass}_R(\text{Ext}_R^i(M, N/I_1^{n_1} \cdots I_t^{n_t} N'))$ and $\text{Ass}_R(\text{Tor}_i^R(M, N/I_1^{n_1} \cdots I_t^{n_t} N'))$ are independent of (n_1, \dots, n_t) for all sufficiently large n_1, \dots, n_t .

Often in mathematics, once an interesting result has been established, other proof of the same result appears. In this article, we give a short proof of a result due to Katz and West.

Let R be a commutative Noetherian ring with identity. Let I be an ideal of R and M a finitely generated R -module. In [6], Ratliff conjectured about the asymptotic behaviour of the set of associated prime ideals $\text{Ass}_R(R/I^n)$ (when R is a domain). Subsequently, Brodmann [1] proved that $\text{Ass}_R(M/I^n M)$ stabilizes for n sufficiently large. Thereafter, this result was extended to an arbitrary finite collection of ideals by Kingsbury and Sharp in [4, Theorem 1.5]; see also [2, Corollary 1.8(c)].

In a different direction, Melkersson and Schenzel generalized Brodmann's result by showing that $\text{Ass}_R(\text{Tor}_i^R(M, R/I^n))$ is independent of n for all large n and for every fixed $i \geq 0$; see [5, Theorem 1]. Recently, in [3, Corollary 3.5], Katz and West proved all the above results in a more general form:

Set-up 1. Let I_1, \dots, I_t be ideals of R . Suppose M and N are finitely generated R -modules and $N' \subseteq N$ a submodule. We set $\mathbb{N} := \{n \in \mathbb{Z} : n \geq 0\}$. Fix $i \in \mathbb{N}$. For every $\mathbf{n} := (n_1, \dots, n_t) \in \mathbb{N}^t$, we denote $\mathbf{I}^{\mathbf{n}} := I_1^{n_1} \cdots I_t^{n_t}$, and we set

$$W_{\mathbf{n}} := \text{Ext}_R^i(M, N/\mathbf{I}^{\mathbf{n}} N') \quad \text{and} \quad W'_{\mathbf{n}} := \text{Tor}_i^R(M, N/\mathbf{I}^{\mathbf{n}} N').$$

Let $W := \bigoplus_{\mathbf{n} \in \mathbb{N}^t} W_{\mathbf{n}}$ and $W' := \bigoplus_{\mathbf{n} \in \mathbb{N}^t} W'_{\mathbf{n}}$.

By the phrase 'for all $\mathbf{n} \gg 0$ ', we mean 'for all $\mathbf{n} = (n_1, \dots, n_t)$ with sufficiently large n_i , $1 \leq i \leq t$ '. With Set-up 1, Katz and West showed that $\text{Ass}_R(W_{\mathbf{n}})$ and $\text{Ass}_R(W'_{\mathbf{n}})$ are independent of \mathbf{n} for all $\mathbf{n} \gg 0$. The aim of this article is to give a short proof of this result. *We prove all the results here for Ext-modules only. For the analogous result of Tor-modules, the proof goes through exactly the same way.*

Discussion 2. For every $1 \leq j \leq t$, \mathbf{e}^j denotes the j th standard basis element of \mathbb{N}^t . Let $\mathcal{R}(\mathbf{I}) := \bigoplus_{\mathbf{n} \in \mathbb{N}^t} \mathbf{I}^{\mathbf{n}}$ be the \mathbb{N}^t -graded Rees ring. The short exact sequence

$$0 \rightarrow \bigoplus_{\mathbf{n} \in \mathbb{N}^t} \mathbf{I}^{\mathbf{n}} N' / \mathbf{I}^{\mathbf{n} + \mathbf{e}^j} N' \rightarrow \bigoplus_{\mathbf{n} \in \mathbb{N}^t} N / \mathbf{I}^{\mathbf{n} + \mathbf{e}^j} N' \rightarrow \bigoplus_{\mathbf{n} \in \mathbb{N}^t} N / \mathbf{I}^{\mathbf{n}} N' \rightarrow 0$$

yields the following exact sequence of \mathbb{N}^t -graded $\mathcal{R}(\mathbf{I})$ -modules:

$$\bigoplus_{\mathbf{n} \in \mathbb{N}^t} \text{Ext}_R^i \left(M, \frac{\mathbf{I}^{\mathbf{n}} N'}{\mathbf{I}^{\mathbf{n} + \mathbf{e}^j} N'} \right) \xrightarrow{\Phi_j} W(\mathbf{e}^j) \rightarrow W \xrightarrow{\Psi_j} \bigoplus_{\mathbf{n} \in \mathbb{N}^t} \text{Ext}_R^{i+1} \left(M, \frac{\mathbf{I}^{\mathbf{n}} N'}{\mathbf{I}^{\mathbf{n} + \mathbf{e}^j} N'} \right),$$

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where $W(\mathbf{e}^j)_{\mathbf{n}} := W_{\mathbf{n}+\mathbf{e}^j}$ for all $\mathbf{n} \in \mathbb{N}^t$. Setting $U^j := \text{Image}(\Phi_j)$ and $V^j := \text{Image}(\Psi_j)$, we obtain the following exact sequence of \mathbb{N}^t -graded $\mathcal{R}(\mathbf{I})$ -modules:

$$(2.1) \quad 0 \longrightarrow U^j \longrightarrow W(\mathbf{e}^j) \longrightarrow W \longrightarrow V^j \longrightarrow 0,$$

where U^j and V^j are finitely generated \mathbb{N}^t -graded $\mathcal{R}(\mathbf{I})$ -modules.

We say an \mathbb{N}^t -graded module E is *eventually zero* (resp. *non-zero*) if $E_{\mathbf{n}} = 0$ for all $\mathbf{n} \gg 0$ (resp. $E_{\mathbf{n}} \neq 0$ for all $\mathbf{n} \gg 0$). By virtue of [7, Theorem 3.4(i)], a finitely generated \mathbb{N}^t -graded $\mathcal{R}(\mathbf{I})$ -module is either eventually zero or eventually non-zero.

Lemma 3. *With the Set-up 1, if (R, \mathfrak{m}, k) is a Noetherian local ring, then each of $\text{Hom}_R(k, W)$ and $\text{Hom}_R(k, W')$ is either eventually zero or eventually non-zero.*

Proof. For every $1 \leq j \leq t$, in view of (2.1), by setting $X^j := \text{Image}(W(\mathbf{e}^j) \rightarrow W)$, we obtain the following short exact sequences of \mathbb{N}^t -graded $\mathcal{R}(\mathbf{I})$ -modules:

$$0 \rightarrow U^j \rightarrow W(\mathbf{e}^j) \rightarrow X^j \rightarrow 0 \quad \text{and} \quad 0 \rightarrow X^j \rightarrow W \rightarrow V^j \rightarrow 0,$$

which induce the following exact sequences of \mathbb{N}^t -graded $\mathcal{R}(\mathbf{I})$ -modules:

$$(3.1) \quad 0 \rightarrow \text{Hom}_R(k, U^j) \rightarrow \text{Hom}_R(k, W(\mathbf{e}^j)) \rightarrow \text{Hom}_R(k, X^j) \rightarrow Y^j \rightarrow 0,$$

$$(3.2) \quad 0 \rightarrow \text{Hom}_R(k, X^j) \rightarrow \text{Hom}_R(k, W) \rightarrow Z^j \rightarrow 0,$$

where Y^j and Z^j (being submodules of $\text{Ext}_R^1(k, U^j)$ and $\text{Hom}_R(k, V^j)$ respectively) are finitely generated \mathbb{N}^t -graded $\mathcal{R}(\mathbf{I})$ -modules. It can be observed from (3.1) and (3.2) that if any one of $\text{Hom}_R(k, U^j)$, Y^j and Z^j ($1 \leq j \leq t$) is eventually non-zero, then so is $\text{Hom}_R(k, W)$, and we are done. So we may assume that $\text{Hom}_R(k, U^j)$, Y^j and Z^j are eventually zero for all $1 \leq j \leq t$. In this case, setting $f(\mathbf{n}) := \text{length}(\text{Hom}_R(k, W_{\mathbf{n}}))$ for all $\mathbf{n} \in \mathbb{N}^t$, in view of the \mathbf{n} th components of (3.1) and (3.2), we obtain that $f(\mathbf{n}+\mathbf{e}^j) = f(\mathbf{n})$ for all $1 \leq j \leq t$ and for all $\mathbf{n} \gg 0$. Therefore $f(\mathbf{n}) = c$ for all $\mathbf{n} \gg 0$, where c is a constant. The lemma now follows easily. \square

Now we can achieve the aim of this article.

Theorem 4. *With the Set-up 1, there exists $\mathbf{k} \in \mathbb{N}^t$ such that the sets $\text{Ass}_R(W_{\mathbf{n}})$ and $\text{Ass}_R(W'_{\mathbf{n}})$ are independent of \mathbf{n} for all $\mathbf{n} \geq \mathbf{k}$.*

Proof. We first show that $\bigcup_{\mathbf{n} \in \mathbb{N}^t} \text{Ass}_R(W_{\mathbf{n}})$ is finite. For every $\mathbf{n} \in \mathbb{N}^t$, the \mathbf{n} th component of the exact sequence (2.1) (for $j = 1$) gives

$$\begin{aligned} \text{Ass}_R(W_{\mathbf{n}+\mathbf{e}^1}) &\subseteq \text{Ass}_R(U_{\mathbf{n}}^1) \cup \text{Ass}_R(W_{\mathbf{n}}) \\ &\subseteq \text{Ass}_R(U_{\mathbf{n}}^1) \cup \text{Ass}_R(U_{\mathbf{n}-\mathbf{e}^1}^1) \cup \text{Ass}_R(W_{\mathbf{n}-\mathbf{e}^1}) \\ &\dots \\ &\subseteq \left(\bigcup_{0 \leq l \leq n_1} \text{Ass}_R(U_{(l, n_2, \dots, n_t)}^1) \right) \cup \text{Ass}_R(W_{(0, n_2, \dots, n_t)}). \end{aligned}$$

Taking union over $\mathbf{n} \in \mathbb{N}^t$, we obtain that

$$\bigcup_{\mathbf{n} \in \mathbb{N}^t} \text{Ass}_R(W_{\mathbf{n}}) \subseteq \left(\bigcup_{\mathbf{n} \in \mathbb{N}^t} \text{Ass}_R(U_{\mathbf{n}}^1) \right) \cup \left(\bigcup_{(n_2, \dots, n_t) \in \mathbb{N}^{t-1}} \text{Ass}_R(W_{(0, n_2, \dots, n_t)}) \right).$$

Since U^1 is finitely generated, the set $\bigcup_{\mathbf{n} \in \mathbb{N}^t} \text{Ass}_R(U_{\mathbf{n}}^1)$ is finite; see [7, Lemma 3.2]. Therefore one obtains that $\bigcup_{\mathbf{n} \in \mathbb{N}^t} \text{Ass}_R(W_{\mathbf{n}})$ is finite by using induction on t .

Since $\bigcup_{\mathbf{n} \in \mathbb{N}^t} \text{Ass}_R(W_{\mathbf{n}})$ is finite, it is now enough to prove that for every $\mathbf{p} \in \bigcup_{\mathbf{n} \in \mathbb{N}^t} \text{Ass}_R(W_{\mathbf{n}})$, exactly one of the following alternatives must hold: either $\mathbf{p} \in$

$\text{Ass}_R(W_{\mathbf{n}})$ for all $\mathbf{n} \gg 0$; or $\mathfrak{p} \notin \text{Ass}_R(W_{\mathbf{n}})$ for all $\mathbf{n} \gg 0$. Localizing at \mathfrak{p} , and replacing $R_{\mathfrak{p}}$ by R and $\mathfrak{p}R_{\mathfrak{p}}$ by \mathfrak{m} , it is now enough to prove that either $\mathfrak{m} \in \text{Ass}_R(W_{\mathbf{n}})$ for all $\mathbf{n} \gg 0$; or $\mathfrak{m} \notin \text{Ass}_R(W_{\mathbf{n}})$ for all $\mathbf{n} \gg 0$, which is equivalent to that either $\text{Hom}_R(k, W_{\mathbf{n}}) \neq 0$ for all $\mathbf{n} \gg 0$; or $\text{Hom}_R(k, W_{\mathbf{n}}) = 0$ for all $\mathbf{n} \gg 0$, where $k := R/\mathfrak{m}$. The last result follows from Lemma 3. \square

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